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Communication

Matroid inequalities

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An important enumerative invariant of a matroid M of rank d is its *h-vector* defined as $(h_0, h_1, \ldots, h_d)$, where h_i is the coefficient of x^{d-i} in the polynomial $T(M; x, 1)$, where $T(M; x, y)$ is the Tutte polynomial of M [3]. We refer to Björner's chapter [1] and to [4] for a comprehensive discussion of the algebraic and topological aspects of this invariant and to [2] for a description of applications to network reliability. The combinatorial significance of the h-vector is due to its direct relation to the vector $(f_0, f_1, \ldots, f_d)$, where f_i is the number of independent sets of cardinality i. This relation is easily seen from the following one-variable specialization of Tutte's famous identity

$$\sum_{i=0}^{d} f_i(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i} = T(M; x, 1).$$

This communication is essentially an announcement of results for h-vectors of matroids obtained in [4], which are stated in the following theorem:

Theorem 1. *The h-vector of a coloop-free rank-d matroid satisfies the following sets of inequalities:*

$$h_0 \leqslant h_1 \leqslant \cdots \leqslant h_{[d/2]}, \tag{1}$$

$$h_i \leqslant h_{d-i}, \quad \text{for } 0 \leqslant i \leqslant [d/2], \tag{2}$$

$$(-1)^j \sum_{i=0}^{j} (-b)^i h_i \geqslant 0, \quad 0 \leqslant j \leqslant d, \tag{3}$$

for any real number $b \geqslant 1$, with equality possible only if $b = 1$.

The proof of inequalities (1) and (2) settles a conjecture of Hibi [5], which is a weaker version of a long-standing conjecture of Stanley [6] that h-vectors of

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matroid complexes are *pure O-sequences* (see [4] for details). The third set of inequalities in Theorem 1 was first shown by Brown and Colbourn [2] using an intricate inductive argument. In [4], two decomposition techniques related to the notion of shellability of simplicial complexes are developed and the combinational consequences of the existence of these decompositions are derived. These general results, when applied to matroid complexes, lead to a proof of Theorem 1. However, in this paper we make no mention of these structural results, nor do we use any terminology from combinatorial topology. Instead, we use elementary properties of the Tutte polynomial to describe an alternative argument that establishes the part of Theorem 1 that pertains to Hibi's conjecture.

In what follows, we will assume familiarity with matroids and their Tutte polynomials [3]. For a coloop-free matroid M of rank d, we can partition the ground set of the matroid into *series classes* since the relation of being (in) a two-element cocircuit is essentially an equivalence relation on the ground set (loops will be considered to be single-element series classes). Now, let X be a series class containing an element e of a coloop-free matroid M. Clearly, if X is independent, then $X - e$ is the set of coloops of the matroid $M - e$. On the other hand if X is dependent, then it is evident that X is a circuit which is disconnected from the rest of the matroid. These two observations lead us to the following lemmas where $t^0 + t^1 + \cdots + t^k$ will be denoted by $u(k;t)$.

Lemma 1. *If a coloop-free matroid has no independent series classes, then it is a direct sum of circuits.*

Lemma 2. *Let X be an independent series class of coloop-free matroid M with $|X| = k + 1$. Then*

$$T(M;x,y) = u(k;x)\,T(M - X;x,y) + T(M/X;x,y).$$

Proof. The proof is by induction on k with the case $k = 0$ being the usual recursion for the Tutte polynomial. The rest of the proof follows easily from the fact that each coloop introduces a multiplicative factor of x in the Tutte polynomial. □

Theorem 2. *If M is a coloop-free matroid, then $T(M;x,1)$ can be written in the following form:*

$$T(M;x,1) = \sum_{i=1}^{n} u(r_{i,1};x)\,u(r_{i,2};x)\cdots u(r_{i,s_i};x), \tag{4}$$

where n is the coefficient h_d of the h-vector $(h_0, h_1, \ldots, h_d)$ of M.

Proof. If all series classes of M are dependent, then, by Lemma 1, M is a direct sum of circuits, and its Tutte polynomial is the product of the Tutte polynomials of these circuits. Since for a circuit C_{k+1} with $k + 1$ elements, we have $T(C_{k+1};x,y) = y + x^1 + x^2 + \cdots + x^k$, the result follows. Now, if M does have an independent series class X, then we apply Lemma 2 to M. Evidently, since X is a series class in the coloop-free matroid M, both $M - X$ and M/X are also coloop-free and hence the result follows by induction on size of the matroid. □

Remarks. Each of the h_d terms in the expression (4) is a polynomial of degree at most d, and, indeed, there is exactly one term of degree exactly d, which corresponds to the fact that $h_0 = f_0 = 1$.

Example. $T(M(K_4); x, y) = x^3 + 3x^2 + 2x + 4xy + 2y + 3y^2 + y^3$ (see [3]) and hence $T(M(K_4); x, 1) = x^3 + 3x^2 + 6x + 6$ which can be written (after following a recursion of the type indicated in the proof of Theorem 2) as follows:

$$T(M(K_4); x, 1) = (1 + x)(1 + x + x^2) + (1 + x) + (1 + x)(1 + x) + (1 + x) + 1 + 1.$$

We will now outline a general argument that shows that the inequalities of Theorem 1 are satisfied by coefficients of any polynomial that is a sum of products of polynomials of the type $u(k; x)$. We define a *P-polynomial* as a polynomial that is a product of polynomials of the form $u(k; x)$ and an *S-polynomial* as a polynomial which is a sum of one or more P-polynomials (not neccesarily of equal degree). A sequence of integers $(g_0, g_1, \ldots, g_d)$ is a rank-d *P-sequence* (*S-sequence*) if there exists a P-polynomial (S-polynomial) $G(x)$ of degree d such that g_i is the coefficient of x^{d-i} in $G(x)$ for every $0 \leqslant i \leqslant d$.

We list below some properties of these special sequences which can be used to prove Theorem 1. The reader can supply the proofs, which are fairly intuitive and straightforward, or refer to [4] where complete proofs are given using a different terminology.

Lemma 3. *A rank-l P-sequence $(g_0, g_1, \ldots, g_l)$ is symmetric and unimodal. In particular, we have*

$$g_i \leqslant g_j \quad \text{whenever } i \leqslant [l/2], \text{ and } i \leqslant j \leqslant l - i.$$

Lemma 4. *Let $(k_0, k_1, \ldots, k_d)$, for $d > l$, denote the sequence $(0, \ldots, 0, g_0, g_1, \ldots, g_l)$, where $(g_0, g_1, \ldots, g_l)$ is a rank-l P-sequence. Then $(k_0, k_1, \ldots, k_d)$ satisfies inequalities* (1) *and* (2).

Since an S-sequence is a sum of sequences of type described in the previous lemma, we immediately have the following:

Proposition 1. *A rank-d S-sequence $(k_0, k_1, \ldots, k_d)$ satisfies inequalities* (1) *and* (2)

The proof of inequalities (1) and (2) for the h-vector of M now follows from Theorem 2 and Proposition 1. With a little more work one can also show that the sequence of Lemma 4 (and hence all S-sequences) satisfy the set of inequalities (3) for the special case of $j = d$ [4]. The proof for any $j \leqslant d$ then follows from the following lemma:

Lemma 5. *If* $(k_0, k_1, \ldots, k_d)$ *is a rank-d S-sequence then* $(k_0, k_1, \ldots, k_j)$ *is a rank-j S-sequence for* $j \leqslant d$.

We should point out that the h-vector inequalities (1) and (2) immediately imply the lower bounds on independence numbers of matroids which were obtained by Björner and Purdy (see [1], Section 7.5). It is conceivable that a further strengthening of such bounds for independence numbers can be obtained by combining all three sets of inequalities. It would also be of interest to see if stronger results can be obtained for connected matroids or minor-closed classes like graphic or cographic matroids. Our results suggest the following question which is likely to be quite difficult but whose solution would be of great significance in this area.

Problem. Characterize the set of S-sequences that can arise as the h-vectors of matroids.

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